Pergamon

Int. J. Solids Structures Vol. 31, No. 4, pp. 497–515, 1994

0020–7683/94 $6.00 + .00

A THREE-DIMENSIONAL ANALYSIS OF ANISOTROPIC INHOMOGENEOUS AND LAMINATED PLATES

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(*Received* 5 *January* 1993; *in revised form* 16 *August* 1993)

Abstract—An asymptotic theory for bending and stretching of anisotropic inhomogeneous and laminated plates is developed based on the three-dimensional elasticity without *a priori* assumptions. The inhomogeneities are considered to vary through the plate thickness, and laminated plates belong to an important class of this inhomogeneous plate. Through appropriate nondimensionalization of the basic equations and expansion of the displacements and stresses in powers of a small parameter, we obtain sets of differential equations of various orders, that can be integrated successively to determine the three-dimensional solutions for the anisotropic inhomogeneous plate under lateral tractions and edge loads. We show that the governing equations for the asymptotic solutions are precisely those in the classical laminated plate theory (CLT) with nonhomogeneous terms. As a result, all the displacement and stress components can be determined in a systematic way, using the same solution method as that for the CLT solution. While the solution is no more difficult than the CLT, the asymptotic solution converges rapidly and gives accurate results. The basic theory and the solution approach are illustrated by considering a problem of symmetric laminated plates under the action of edge loads, and Pagano's problem of a bi-directional laminated plate under lateral transverse loads. Elasticity solutions for the problems are obtained in a simple manner, without treating the individual layers and considering the interfacial continuity conditions in particular.

1. INTRODUCTION

Analysis of bending and stretching of laminated elastic plates is an important class of problems in mechanics of composite materials. There are a large number of publications on the subject. An assessment of the different approaches used for modeling laminated plates was given in a review paper (Noor and Burton, 1989). Many of the theories developed for laminated plates were based on the extension of the classical theory of homogeneous plates, for example, the classical laminated plate theory (CLT) [see, e.g. Ashton and Whitney (1970) and Christensen (1979)] and various higher-order theories (Lo *et al.*, 1977a, b, 1978). These theories made *a priori* assumptions regarding the variation of displacements in the thickness direction. As a result, they provide little information about the error estimation in the response predictions. To verify a proposed model, usually the numerical results obtained were compared with Pagano's elasticity solution (Pagano, 1969, 1970) for rectangular multi-layered bi-directional composite plates with simple support conditions, which has been regarded as a benchmark in studying bending of laminated plates. It should be noted that Pagano's solution was obtained for a specific problem. Be it analytical, it requires the solution of $6N$ (N is the number of layers) simultaneous algebraic equations resulting from the continuity conditions imposed on the interfacial displacements and tractions.

Recently, Basi *et al.* (1991) proposed a theory for stress analysis of anisotropic inhomogeneous plates in which the elastic moduli vary through the thickness. The laminated plates may be considered as inhomogeneous plates with piecewise constant moduli through the thickness. Earlier work of their studies (Rogers and Spencer, 1989) was concentrated on inhomogeneous isotropic plates. A dominant feature of the theory is that the solution, in case of inhomogeneities that are symmetric about the midplane of the plate, may be expressed in terms of the solution for an equivalent homogeneous plate. In their analysis attention was restricted to symmetric plates under edge loads. Therefore, the advantage of complete uncoupling of bending and stretching of the plate was fully employed. Furthermore, their formulation was limited to the condition of zero traction on the lateral surfaces of the plate. The important problems of bending and stretching due to lateral loads were not considered.

In this paper, we first recast the full three-dimensional elasticity equations for the anisotropic inhomogeneous plates in such forms that are expressed in terms of differential operators involving the thickness coordinate and the in-plane coordinates, respectively. A close examination of the length scales of the plate dimensions and those of the displacement and stress components reveals that it is constructive to introduce appropriate dimensionless quantities in the equations. The dimensionless form of the equations suggests that the method of asymptotic expansion is well suited for the problem. By expanding the displacement and stress components in terms of powers of ε^2, where $\varepsilon = h/L$, $2h$ is the plate thickness and L is a typical in-plane plate dimension, it becomes apparent that the three-dimensional equations are reduced to recurrent sets of differential equations that can be integrated in turn with respect to the thickness coordinate to obtain the asymptotic solution for the problem. While it is simple to find in the integration process that the leading order displacement field is just the displacements according to Kirchhoff's thin-plate assumptions, it turns out that the leading-order equations for the displacements are precisely the CLT equations. Moreover, in the asymptotic solutions that follow, the same CLT equations with nonhomogeneous terms that are completely determined by the preceding step are obtained. At each level of the asymptotic solution all the relevant quantities of displacements and stresses, including in-plane stresses and transverse shear and normal stresses, can be determined in a systematic way no more difficult than the corresponding CLT solution. The reduction to CLT equations is a natural consequence of the asymptotic integration process without prior assumptions in that we do not restrict attention to symmetric plates under edge loads. At each level of approximation the recurrent equations and the constitutive equations for the anisotropic inhomogeneous materials, together with the prescribed loading condition on the lateral surfaces of the plate are exactly satisfied. The edge conditions are clearly specified as well. In the case of laminated plates, the theory is applicable regardless of the number of layers as long as $\varepsilon < 1$. There is no need to treat the interfacial continuity in particular; the conditions of tractions and displacement continuity are inherently satisfied. Extension of the theory to include hygrothermal effect and to consider other structural responses can be made. Furthermore, the order of the solution being sought provides a quick estimation of the accuracy of the results. When the plate is deformed by the edge loads in the absence of the lateral load, carrying out only a single step of the solution, in general, gives results that are accurate to $O(\varepsilon^2)$ for the in-plane stresses, $O(\varepsilon^3)$ for the transverse shear stresses, and $O(\varepsilon^4)$ for the transverse normal stress. When the lateral load is present, carrying out two steps of the solution will yield results with the same degree of accuracy. A further solution step in both cases will increase the accuracy according to the order of ε^2 for all the stress components.

In Section 2 we first recast the three-dimensional elasticity equations and the boundary conditions for a general problem into a form convenient for the subsequent analysis. The dimensionless forms of the equations and the recurrent relations are introduced in Section 3. Successive integration of the differential equations and determination of the integration functions, leading to the equivalent CLT equations, are detailed in Section 4. The inhomogeneities through the thickness of anisotropic laminated plates are considered in Section 5. In Section 6 the general solution for the resulting two-dimensional equations in the case of symmetric plates is given. Finally, to illustrate the basic theory and the solution approach, we present in Section 7 the analytical solutions for anisotropic inhomogeneous and laminated plates under the action of tractions and moment distributed uniformly along the edges, and treat Pagano's problem in the context of the inhomogeneous plate. The elasticity solution for the problem is obtained in a simple manner according to the present theory.

2. THE BASIC EQUATIONS

Consider an anisotropic inhomogeneous plate having in each point one plane of elastic symmetry parallel to the midplane. The plate is of uniform thickness $2h$. Let the axis x_3 be directed downward from the origin of the Cartesian coordinates x_1, x_2, x_3 and the midplane coincide with the plane $x_3 = 0$. Thus, through the thickness $-h < x_3 < h$. On the lateral surface $x_3 = -h$ the transvese load $q(x_1, x_2)$ is prescribed. On the surface $x_3 = h$ the plate is free from external loads. Along the edges of the plate, appropriate edge boundary conditions are prescribed.

The stress–displacement constitutive relations are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{1,2}+u_{2,1} \end{Bmatrix}, \tag{1}$$

where the displacement components are denoted by u_1, u_2 and u_3; the commas denote partial differentiation with respect to the suffix variables. $\sigma_{11}, \sigma_{22}, \ldots$ and σ_{12} are the stress components c_{ij} $(i, j = 1, 2, \ldots, 6)$ are the 13 elastic constants of the anisotropic material with one plane of symmetry.

The material is assumed to be inhomogeneous through the plate thickness. Thus, the material properties are dependent on x_3, i.e. $c_{ij} = c_{ij}(x_3)$. When the plate is symmetric about the midplane, $c_{ij}(x_3) = c_{ij}(-x_3)$. This is advantageous in obtaining the solution for a problem but not required in formulating the theory.

The equilibrium equations without body forces are

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0, \tag{2}$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0, \tag{3}$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0. \tag{4}$$

Expressing the stresses in terms of the differential operators with respect to u_1, u_2 and u_3, we may rewrite (1) in the following form:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11}\frac{\partial}{\partial x_1} + c_{16}\frac{\partial}{\partial x_2} & c_{16}\frac{\partial}{\partial x_1} + c_{12}\frac{\partial}{\partial x_2} & c_{13}\frac{\partial}{\partial x_3} \\ c_{12}\frac{\partial}{\partial x_1} + c_{26}\frac{\partial}{\partial x_2} & c_{26}\frac{\partial}{\partial x_1} + c_{22}\frac{\partial}{\partial x_2} & c_{23}\frac{\partial}{\partial x_3} \\ c_{13}\frac{\partial}{\partial x_1} + c_{36}\frac{\partial}{\partial x_2} & c_{36}\frac{\partial}{\partial x_1} + c_{23}\frac{\partial}{\partial x_2} & c_{33}\frac{\partial}{\partial x_3} \\ c_{45}\frac{\partial}{\partial x_3} & c_{44}\frac{\partial}{\partial x_3} & c_{45}\frac{\partial}{\partial x_1} + c_{44}\frac{\partial}{\partial x_2} \\ c_{55}\frac{\partial}{\partial x_3} & c_{45}\frac{\partial}{\partial x_3} & c_{55}\frac{\partial}{\partial x_1} + c_{45}\frac{\partial}{\partial x_2} \\ c_{16}\frac{\partial}{\partial x_1} + c_{66}\frac{\partial}{\partial x_2} & c_{66}\frac{\partial}{\partial x_1} + c_{26}\frac{\partial}{\partial x_2} & c_{36}\frac{\partial}{\partial x_3} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}. \tag{5}$$

Eliminating σ_{11}, σ_{22} and σ_{12} from (2), (3) and (5), we obtain the following equations with u_1, u_2, u_3 and σ_{13}, σ_{23}, σ_{33} as the field variables:

$$u_{3,3} = -[L_{13} \quad L_{23}]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + L_{33}\sigma_{33}, \tag{6}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{,3} = -\begin{bmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} \end{bmatrix}^{\mathrm{T}} u_3 + \begin{bmatrix} s_{55} & s_{45} \\ s_{45} & s_{44} \end{bmatrix}\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix}, \tag{7}$$

$$\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix}_{,3} = -\begin{bmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} L_{13} \\ L_{23} \end{Bmatrix}\sigma_{33}, \tag{8}$$

$$\sigma_{33,3} = -\begin{bmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} \end{bmatrix}\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix}, \tag{9}$$

where the superscript T denotes the transpose, and

$$L_{11} = \left[Q_{11}\frac{\partial^2}{\partial x_1^2} + 2Q_{16}\frac{\partial^2}{\partial x_1 \partial x_2} + Q_{66}\frac{\partial^2}{\partial x_2^2}\right],$$

$$L_{12} = \left[Q_{16}\frac{\partial^2}{\partial x_1^2} + (Q_{12}+Q_{66})\frac{\partial^2}{\partial x_1 \partial x_2} + Q_{26}\frac{\partial^2}{\partial x_2^2}\right],$$

$$L_{22} = \left[Q_{66}\frac{\partial^2}{\partial x_1^2} + 2Q_{26}\frac{\partial^2}{\partial x_1 \partial x_2} + Q_{22}\frac{\partial^2}{\partial x_2^2}\right],$$

$$L_{13} = \left(c_{13}\frac{\partial}{\partial x_1} + c_{36}\frac{\partial}{\partial x_2}\right)\Big/ c_{33},$$

$$L_{23} = \left(c_{36}\frac{\partial}{\partial x_1} + c_{23}\frac{\partial}{\partial x_2}\right)\Big/ c_{33}, \quad L_{33} = 1/c_{33},$$

$$\begin{bmatrix} s_{55} & s_{45} \\ s_{45} & s_{44} \end{bmatrix} = \begin{bmatrix} c_{55} & c_{45} \\ c_{45} & c_{44} \end{bmatrix}^{-1}, \quad Q_{ij} = c_{ij} - c_{i3}c_{j3}/c_{33}, \quad (i,j = 1,2,6).$$

Notice that (6)–(9) have been cast in such forms that on the left-hand side the differentiations are with respect to x_3, whereas on the right-hand side the differential operators are all expressed in terms of x_1 and x_2. The in-plane stresses σ_{11}, σ_{22} and σ_{12} expressed in terms of u_1, u_2 and σ_{33} take the form

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11}\dfrac{\partial}{\partial x_1} + Q_{16}\dfrac{\partial}{\partial x_2} & Q_{16}\dfrac{\partial}{\partial x_1} + Q_{12}\dfrac{\partial}{\partial x_2} \\ Q_{12}\dfrac{\partial}{\partial x_1} + Q_{26}\dfrac{\partial}{\partial x_2} & Q_{26}\dfrac{\partial}{\partial x_1} + Q_{22}\dfrac{\partial}{\partial x_2} \\ Q_{16}\dfrac{\partial}{\partial x_1} + Q_{66}\dfrac{\partial}{\partial x_2} & Q_{66}\dfrac{\partial}{\partial x_1} + Q_{26}\dfrac{\partial}{\partial x_2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{Bmatrix} c_{13}/c_{33} \\ c_{23}/c_{33} \\ c_{36}/c_{33} \end{Bmatrix}\sigma_{33}. \tag{10}$$

Boundary conditions of the problem are specified as follows:

On the lateral surface the transverse load $q(x_1, x_2)$ is prescribed

$$\begin{aligned} [\sigma_{13} \quad \sigma_{23}] &= [0, 0] && \text{on } x_3 = \pm h, \\ \sigma_{33} &= -q(x_1, x_2) && \text{on } x_3 = -h, \\ \sigma_{33} &= 0 && \text{on } x_3 = h. \end{aligned} \tag{11}$$

Along the edges Γ_σ tractions $p_1, p_2\, p_3$ are prescribed

$$\begin{bmatrix} n & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix},$$

$$\text{on } \Gamma_\sigma, \tag{12}$$

$$[n_1 \quad n_2] \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = p_3,$$

in which n_1, n_2 denote the outward unit normal at a point along the edge.

Along the edges Γ_u displacements u_1^0, u_2^0, u_3^0 are prescribed,

$$u_1 = u_1^0, \quad u_2 = u_{32}^0, \quad u_3 = u_3^0, \quad \text{on } \Gamma_u. \tag{13}$$

Note that the boundary conditions are deliberately written in matrix form to facilitate the analysis that follows.

3. NONDIMENSIONALIZATION AND RECURRENCE EQUATIONS

The plate thickness is small in comparison with the in-plane plate dimensions. Because of the length scales of the plate, we introduce the following dimensionless coordinates:

$$x = x_1/L, \quad y = x_2/L, \quad z = x_3/h, \tag{14}$$

where L denotes a typical in-plane dimension of the plate, and $-1 < z < 1$.

In general, displacement w is increasing with the plate dimensions and is relatively larger than the in-plane deformation for bending and stretching response of the plate. Further, the stresses σ_{13}, σ_{23} and σ_{33} are of secondary quantities compared with the in-plane stresses σ_{11}, σ_{22} and σ_{12}. A close examination of the dimensions in (6)–(10) reveals that it is constructive to introduce in the formulation the following dimensionless quantities:

$$u = u_1/h, \quad v = u_2/h, \quad w = u_3/L, \tag{15}$$

and

$$\begin{aligned} \sigma_x &= \sigma_{11}/Q\varepsilon, \quad & \sigma_y &= \sigma_{22}/Q\varepsilon, \quad & \sigma_{xy} &= \sigma_{12}/Q\varepsilon, \\ \sigma_{xz} &= \sigma_{13}/Q\varepsilon^2, \quad & \sigma_{yz} &= \sigma_{23}/Q\varepsilon^2, \quad & \sigma_z &= \sigma_{33}/Q\varepsilon^3, \end{aligned} \tag{16}$$

in which u, v and w are the dimensionless displacements; σ_x, σ_y, σ_z, σ_{xz}, σ_{yz} and σ_{xy} are the dimensionless stresses; $\varepsilon = h/L < 1$ is a dimensionless parameter. An elastic modulus or a reference uniform load that has the dimension of the elastic constants can be chosen as Q. To be specific, let $Q = c_{33}$.

Upon introducing (14)–(16) in (6)–(10), the dimensionless forms of the three-dimensional elasticity equations for the plate can be written as

$$w_{,z} = -\varepsilon^2 [l_{13} \quad l_{23}] \begin{Bmatrix} u \\ v \end{Bmatrix} + \varepsilon^4 \sigma_z, \tag{17}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix}_{,z} = -\begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix}^{\mathrm{T}} w + \varepsilon^2 \begin{bmatrix} \tilde{s}_{55} & \tilde{s}_{45} \\ \tilde{s}_{45} & \tilde{s}_{44} \end{bmatrix} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}, \tag{18}$$

$$\begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}_{,z} = -\begin{bmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} - \varepsilon^2 \begin{Bmatrix} l_{13} \\ l_{23} \end{Bmatrix} \sigma_z, \tag{19}$$

$$\sigma_{z,z} = -\begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}, \tag{20}$$

and

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} l_{14} & l_{24} \\ l_{15} & l_{25} \\ l_{16} & l_{26} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} l_{34} \\ l_{35} \\ l_{36} \end{Bmatrix} \sigma_z, \tag{21}$$

where

$$l_{11} = \left[\tilde{Q}_{11}\frac{\partial^2}{\partial x^2} + 2\tilde{Q}_{16}\frac{\partial^2}{\partial x\,\partial y} + \tilde{Q}_{66}\frac{\partial^2}{\partial y^2}\right],$$

$$l_{12} = \left[\tilde{Q}_{16}\frac{\partial^2}{\partial x^2} + (\tilde{Q}_{12} + \tilde{Q}_{66})\frac{\partial^2}{\partial x\,\partial y} + \tilde{Q}_{26}\frac{\partial^2}{\partial y^2}\right],$$

$$l_{22} = \left[\tilde{Q}_{66}\frac{\partial^2}{\partial x^2} + 2\tilde{Q}_{26}\frac{\partial^2}{\partial x\,\partial y} + \tilde{Q}_{22}\frac{\partial^2}{\partial y^2}\right],$$

$$l_{13} = \left(c_{13}\frac{\partial}{\partial x} + c_{36}\frac{\partial}{\partial y}\right)\Big/ c_{33}, \quad l_{23} = \left(c_{36}\frac{\partial}{\partial x} + c_{23}\frac{\partial}{\partial y}\right)\Big/ c_{33},$$

$$l_{14} = \left(\tilde{Q}_{11}\frac{\partial}{\partial x} + \tilde{Q}_{16}\frac{\partial}{\partial y}\right), \quad l_{24} = \left(\tilde{Q}_{16}\frac{\partial}{\partial x} + \tilde{Q}_{12}\frac{\partial}{\partial y}\right),$$

$$l_{15} = \left(\tilde{Q}_{12}\frac{\partial}{\partial x} + \tilde{Q}_{26}\frac{\partial}{\partial y}\right), \quad l_{25} = \left(\tilde{Q}_{26}\frac{\partial}{\partial x} + \tilde{Q}_{22}\frac{\partial}{\partial y}\right),$$

$$l_{16} = \left(\tilde{Q}_{16}\frac{\partial}{\partial x} + \tilde{Q}_{66}\frac{\partial}{\partial y}\right), \quad l_{26} = \left(\tilde{Q}_{66}\frac{\partial}{\partial x} + \tilde{Q}_{26}\frac{\partial}{\partial y}\right),$$

$$l_{34} = c_{13}/c_{33}, \quad l_{35} = c_{23}/c_{33}, \quad l_{36} = c_{36}/c_{33},$$

$$\tilde{Q}_{ij} = Q_{ij}/c_{33} \quad (i, j = 1, 2, 6), \quad \tilde{s}_{ij} = s_{ij}c_{33} \quad (i, j = 4, 5).$$

Note that (17)–(21) contain only terms involving powers of ε^0, ε^2 and ε^4. Therefore, we may expand the displacements and all the stresses in terms of powers of ε^2 as given by

$$f(x, y, z; \varepsilon) = f_{(0)}(x, y, z) + \varepsilon^2 f_{(1)}(x, y, z) + \varepsilon^4 f_{(2)}(x, y, z) + \cdots. \tag{22}$$

Upon substituting (22) into (17)–(21) and collecting coefficients of equal powers of ε, we obtain the following sets of equations:

Order ε^0:

$$w_{(0),z} = 0, \tag{23}$$

$$\mathbf{u}_{(0),z} = -\mathbf{D}^{\mathrm{T}} w_{(0)}, \tag{24}$$

$$\sigma_{s(0),z} = -\mathbf{L}_1 \mathbf{u}_{(0)}, \tag{25}$$

$$\sigma_{z(0),z} = -\mathbf{D}\sigma_{s(0)} \tag{26}$$

and

$$\sigma_{p(0)} = \mathbf{L}_3\mathbf{u}_{(0)}. \tag{27}$$

Order ε^2:

$$w_{(1),z} = -\mathbf{L}_2\mathbf{u}_{(0)}, \tag{28}$$

$$\mathbf{u}_{(1),z} = -\mathbf{D}^{\mathrm{T}}w_{(1)} + \mathbf{S}\sigma_{s(0)}, \tag{29}$$

$$\sigma_{s(1),z} = -\mathbf{L}_1\mathbf{u}_{(1)} - \mathbf{L}_2^{\mathrm{T}}\sigma_{z(0)}, \tag{30}$$

$$\sigma_{z(1),z} = -\mathbf{D}\sigma_{s(1)} \tag{31}$$

and

$$\sigma_{p(1)} = \mathbf{L}_3\mathbf{u}_{(1)} + \mathbf{L}_4\sigma_{z(0)}. \tag{32}$$

Order ε^{2k} $(k = 2, 3, \ldots)$:

$$w_{(k),z} = -\mathbf{L}_2\mathbf{u}_{(k-1)} + \sigma_{z(k-2)}, \tag{33}$$

$$\mathbf{u}_{(k),z} = -\mathbf{D}^{\mathrm{T}}w_{(k)} + \mathbf{S}\sigma_{s(k-1)}, \tag{34}$$

$$\sigma_{s(k),z} = -\mathbf{L}_1\mathbf{u}_{(k)} - \mathbf{L}_2^{\mathrm{T}}\sigma_{z(k-1)}, \tag{35}$$

$$\sigma_{z(k),z} = -\mathbf{D}\sigma_{s(k)} \tag{36}$$

and

$$\sigma_{p(k)} = \mathbf{L}_3\mathbf{u}_{(k)} + \mathbf{L}_4\sigma_{z(k-1)}, \tag{37}$$

where

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \sigma_s = \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}, \quad \sigma_p = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{bmatrix},$$

$$\mathbf{L}_2 = [l_{13} \quad l_{23}], \quad \mathbf{L}_3 = \begin{bmatrix} l_{14} & l_{24} \\ l_{15} & l_{25} \\ l_{16} & l_{26} \end{bmatrix}, \quad \mathbf{L}_4 = \begin{bmatrix} l_{34} \\ l_{35} \\ l_{36} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \tilde{s}_{55} & \tilde{s}_{45} \\ \tilde{s}_{45} & \tilde{s}_{44} \end{bmatrix}.$$

The associated dimensionless boundary conditions are given by:

Order ε^0:

$$[\sigma_{xz} \quad \sigma_{yz}]_{(0)} = [0, \quad 0] \quad \text{on } z = \pm 1, \tag{38}$$

$$\sigma_{z(0)} = -\tilde{q}(x, y) \quad \text{on } z = -1, \tag{39}$$

$$\sigma_{z(0)} = 0 \quad \text{on } z = 1, \tag{40}$$

$$\begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix} \mathbf{L}_3\mathbf{u}_{(0)} = \begin{Bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{Bmatrix} \tag{41}$$

$$\text{on } \Gamma_\sigma, \tag{42}$$

$$[n_1 \quad n_2] \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}_{(0)} = \tilde{p}_3$$

$$u_{(0)} = u^0, \quad v_{(0)} = v^0, \quad w_{(0)} = w^0 \quad \text{on } \Gamma_u. \tag{43}$$

Order ε^{2k} $(k = 1, 2, 3, \ldots)$:

$$[\sigma_{xz} \quad \sigma_{yz}]_{(k)} = [0, \quad 0] \quad \text{on } z = \pm 1, \tag{44}$$

$$\sigma_{z(k)} = 0 \quad \text{on } z = \pm 1, \tag{45}$$

$$\begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix} (\mathbf{L}_3 \mathbf{u}_{(k)} + \mathbf{L}_4 \sigma_{z(k-1)}) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{46}$$

on Γ_σ,

$$[n_1 \quad n_2] \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}_{(k)} = 0 \tag{47}$$

$$u_{(k)} = v_{(k)} = w_{(k)} = 0 \quad \text{on } \Gamma_u, \tag{48}$$

where $\tilde{q} = q/Q\varepsilon^3$, $\tilde{p}_k = p_k/Q\varepsilon$, $(k = 1, 2)$, $\tilde{p}_3 = p_3/Q\varepsilon^2$, $u^0 = u_1^0/h$, $v^0 = u_2^0/h$, $w^0 = u_3^0/L$.

4. ASYMPTOTIC SOLUTION AND CLT

The recurrent relations suggest that the solution for the problem can be determined by integrating the differential equations with respect to z in turn. The associated boundary conditions, of course, must be satisfied at each level of the asymptotic solution. Thus, we obtain from (23) that

$$w_{(0)} = w_0(x, y), \tag{49}$$

where $w_0(x, y)$ is an integration function to be determined from the boundary conditions.

From (23) we find

$$\mathbf{u}_{(0)} = -z\mathbf{D}^{\mathrm{T}} w_0 + \mathbf{u}_0, \tag{50}$$

where $\mathbf{u}_0 = [u_0, v_0]^{\mathrm{T}}$ are integration functions of x and y to be determined from the boundary conditions.

We remark in passing that (50) is just the displacements according to Kirchhoff's thin plate assumptions.

Integrating (25), (26), we obtain

$$\sigma_{s(0)} = \int_{-1}^{z} \mathbf{L}_1 (z\mathbf{D}^{\mathrm{T}} w_0 - \mathbf{u}_0)\, \mathrm{d}z, \tag{51}$$

$$\begin{aligned} \sigma_{z(0)} &= \int_{-1}^{z} -\mathbf{D} \left\{ \int_{-1}^{z} \mathbf{L}_1 (z\mathbf{D}^{\mathrm{T}} w_0 - \mathbf{u}_0)\, \mathrm{d}z \right\} \mathrm{d}z - \tilde{q}(x, y) \\ &= -\int_{-1}^{z} (z - \eta) \mathbf{D} \mathbf{L}_1 (\eta \mathbf{D}^{\mathrm{T}} w_0 - \mathbf{u}_0)\, \mathrm{d}\eta - \tilde{q}(x, y), \end{aligned} \tag{52}$$

in (52) we have used the integration by parts to reduce the double integral to the form of a single integral, and $l_{ij} = l_{ij}(\eta)$, $(i, j = 1, 2)$ in $\mathbf{L}_1$.

Consideration of the boundary conditions is now in order. On $z = -1$, the lateral conditions are identically satisfied by (51) and (52). Whereas the boundary condition $[\sigma_{xz}, \sigma_{yz}]_{(0)} = [0, 0]$ on $z = 1$ gives

$$\int_{-1}^{1} \mathbf{L}_1(\mathbf{u}_0 - z\mathbf{D}^{\mathrm{T}} w_0)\,\mathrm{d}z = \mathbf{0}. \tag{53}$$

Carrying out the simple operation in (53), we obtain

$$\left(A_{11}\frac{\partial^2}{\partial x^2} + 2A_{16}\frac{\partial^2}{\partial x\,\partial y} + A_{66}\frac{\partial^2}{\partial y^2}\right)u_0 + \left[A_{16}\frac{\partial^2}{\partial x^2} + (A_{12}+A_{66})\frac{\partial^2}{\partial x\,\partial y} + A_{26}\frac{\partial^2}{\partial y^2}\right]v_0$$
$$- \left[B_{11}\frac{\partial^3}{\partial x^3} + 3B_{16}\frac{\partial^3}{\partial x^2\,\partial y} + (B_{12}+2B_{66})\frac{\partial^3}{\partial x\,\partial y^2} + B_{26}\frac{\partial^3}{\partial y^3}\right]w_0 = 0, \tag{54}$$

$$\left[A_{16}\frac{\partial^2}{\partial x^2} + (A_{12}+A_{66})\frac{\partial^2}{\partial x\,\partial y} + A_{26}\frac{\partial^2}{\partial y^2}\right]u_0 + \left(A_{66}\frac{\partial^2}{\partial x^2} + 2A_{26}\frac{\partial^2}{\partial x\,\partial y} + A_{22}\frac{\partial^2}{\partial y^2}\right)v_0$$
$$- \left[B_{16}\frac{\partial^3}{\partial x^3} + (B_{12}+2B_{66})\frac{\partial^3}{\partial x^2\,\partial y} + 3B_{26}\frac{\partial^3}{\partial x\,\partial y^2} + B_{22}\frac{\partial^3}{\partial y^3}\right]w_0 = 0, \tag{55}$$

where $A_{ij} = \int_{-1}^{1} \tilde{Q}_{ij}\,\mathrm{d}z$, $B_{ij} = \int_{-1}^{1} \tilde{Q}_{ij} z\,\mathrm{d}z$, $(i, j = 1, 2, 6)$.

The boundary condition $\sigma_{z(0)} = 0$ on $z = 1$ gives

$$\int_{-1}^{1} (1-z)\mathbf{D}\mathbf{L}_1(z\mathbf{D}^{\mathrm{T}} w_0 - \mathbf{u}_0)\,\mathrm{d}z = -\tilde{q}(x, y). \tag{56}$$

Upon using (53) in (56), this equation can be written explicitly as

$$\left[D_{11}\frac{\partial^4}{\partial x^4} + 4D_{16}\frac{\partial^4}{\partial x^3\,\partial y} + 2(D_{12}+2D_{66})\frac{\partial^4}{\partial x^2\,\partial y^2} + 4D_{26}\frac{\partial^4}{\partial x\,\partial y^3} + D_{22}\frac{\partial^4}{\partial y^4}\right]w_0$$
$$- \left[B_{11}\frac{\partial^3}{\partial x^3} + 3B_{16}\frac{\partial^3}{\partial x^2\,\partial y} + (B_{12}+2B_{66})\frac{\partial^3}{\partial x\,\partial y^2} + B_{26}\frac{\partial^3}{\partial y^3}\right]u_0$$
$$- \left[B_{16}\frac{\partial^3}{\partial x^3} + (B_{12}+2B_{66})\frac{\partial^3}{\partial x^2\,\partial y} + 3B_{26}\frac{\partial^3}{\partial x\,\partial y^2} + B_{22}\frac{\partial^3}{\partial y^3}\right]v_0 = \tilde{q}(x, y), \tag{57}$$

where $D_{ij} = \int_{-1}^{1} \tilde{Q}_{ij} z^2\,\mathrm{d}z\,(i, j = 1, 2, 6)$.

Equations (54), (55) and (57) are precisely the governing equations for the displacements in the classical laminated plate theory. Therefore, the equations derived from order ε^0 for an anisotropic *inhomogeneous* plate are the CLT governing equations for an equivalent anisotropic *homogeneous* plate. Solution of (54), (55) and (57) must be supplemented with the associated edge boundary conditions (41)–(43). For the order ε^0 solution, once u_0, v_0 and w_0 are determined, the displacements are given by (49), (50), the transverse stresses by (51), (52), and the in-plane stresses by (27).

Carrying on the solution to order ε^2, following the same line as was done in order ε^0, we obtain

$$w_{(1)} = w_1(x, y) + \varphi_1(x, y, z), \tag{58}$$

$$\mathbf{u}_{(1)} = -z\mathbf{D}^{\mathrm{T}} w_1 + \mathbf{u}_1 + \boldsymbol{\phi}_1, \tag{59}$$

$$\sigma_{s(1)} = \int_{-1}^{z} \mathbf{L}_1(z\mathbf{D}^{\mathrm{T}}w_1 - \mathbf{u}_1)\,\mathrm{d}z + \mathbf{f}_1, \tag{60}$$

$$\sigma_{z(1)} = -\int_{-1}^{z} (z-\eta)\mathbf{D}\mathbf{L}_1(\eta\mathbf{D}^{\mathrm{T}}w_1 - \mathbf{u}_1)\,\mathrm{d}\eta + q_1, \tag{61}$$

in which

$$\varphi_1(x,y,z) = -\int_{-1}^{z} \mathbf{L}_2\mathbf{u}_{(0)}\,\mathrm{d}\eta,$$

$$\mathbf{u}_1 = [u_1(x,y) \quad v_1(x,y)]^{\mathrm{T}},$$

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} \phi_{11}(x,y,z) \\ \phi_{21}(x,y,z) \end{Bmatrix} = \int_{-1}^{z} (\mathbf{S}\boldsymbol{\sigma}_{s(0)} - \mathbf{D}^{\mathrm{T}}\varphi_1)\,\mathrm{d}\eta,$$

$$\mathbf{f}_1 = \begin{Bmatrix} f_{11}(x,y,z) \\ f_{21}(x,y,z) \end{Bmatrix} = -\int_{-1}^{z} (\mathbf{L}_1\boldsymbol{\phi}_1 + \mathbf{L}_2^{\mathrm{T}}\sigma_{z(0)})\,\mathrm{d}\eta,$$

$$q_1(x,y,z) = \int_{-1}^{z} (z-\eta)\mathbf{D}(\mathbf{L}_1\boldsymbol{\phi}_1 + \mathbf{L}_2^{\mathrm{T}}\sigma_{z(0)})\,\mathrm{d}\eta.$$

The functions $u_1(x,y)$, $v_1(x,y)$ and $w_1(x,y)$ deduced from the integration are again determined by the boundary conditions specified by (44)–(48). It is easily seen that the boundary conditions on $z = -1$ are identically satisfied, and the boundary condition $[\sigma_{xz}, \sigma_{yz}]_{(1)} = [0, 0]$ on $z = 1$ leads to the first two CLT equations with nonhomogeneous terms $f_{11}(x,y,1)$ and $f_{21}(x,y,1)$ presented, respectively, on the right-hand side of the equations (54) and (55), in which the dependent field variables are changed to $u_1(x,y)$, $v_1(x,y)$ and $w_1(x,y)$.

Using (61) in the condition $\sigma_{z(1)} = 0$ on $z = 1$ leads to the third CLT equation with

$$\tilde{q}_1(x,y) = \mathbf{D}\int_{-1}^{1} z(\mathbf{L}_1\boldsymbol{\phi}_1 + \mathbf{L}_2^{\mathrm{T}}\sigma_{z(0)})\,\mathrm{d}z$$

as the nonhomogeneous term in place of $\tilde{q}(x,y)$ in (57). The edge conditions for the ε^2 equations are given by (47) and (48). The in-plane stresses for order ε^2 are obtained from (32).

For order ε^{2k} $(k = 2, 3, \ldots)$, the solution can be determined in a similar fashion by integrating the higher-order equations. The governing equations for $u_k(x,y)$, $v_k(x,y)$ and $w_k(x,y)$ are again of the same forms as the CLT equations, with

$$\varphi_k(x,y,z) = -\int_{-1}^{z} (\mathbf{L}_2 u_{(k-1)} - \sigma_{z(k-2)})\,\mathrm{d}z$$

replacing $\varphi_1(x,y,z)$ in the corresponding ε^2-order equations and in the edge boundary conditions.

A cursory examination of the asymptotic expressions (22) reveals that the solution is convergent according to the powers of ε^2. Since the dimensionless edge tractions are of the order ε^{-1}, the dimensionless lateral loads are of order ε^{-3}, taking account of the order in the dimensionless stresses, we may conclude that the ε^0 solution, in the case where only edge loads are applied to the plate, provides the transverse shear stresses σ_{13} and σ_{23} that are accurate to $O(\varepsilon^3)$, the transverse normal stress σ_{33} accurate to $O(\varepsilon^4)$, and the in-plane stresses σ_{11}, σ_{22} and σ_{12} accurate to $O(\varepsilon^2)$. When the lateral load is present, the asymptotic solution must be determined to the next-order correction in order to obtain the same degree of accuracy. Nevertheless, it takes only two solution steps in this case and each solution is no more difficult than the corresponding CLT solution for the problem.

5. LAMINATED PLATES

The laminated plate is usually composed of composite laminae stacked together with each lamina oriented in a different direction relative to the axes of the laminate. We now consider a laminated plate consisting of N layers and the individual lamina material is anisotropic with one plane of elastic symmetry coincident with the coordinate plane x_1–x_2. Obviously, the laminated plate is an anisotropic inhomogeneous plate with piecewise constant elastic moduli through the thickness.

The stress–strain relations in (1) are expressed in the chosen axes of the plate. Therefore, the elastic constants of each layer must be transformed from its principal material axes to the laminate axes in order to obtain $c_{ij}(x_3)$ for the plate. The transformation relations can be found in many books [see, e.g. Christensen (1979)]. There are 13 nonzero quantities expressed in terms of the orientation angles and the elastic constants for each layer of the inhomogeneous plate.

Let us denote the c_{ij} in the laminate axes for the kth layer by $c_{ij}^{(k)}$, the thickness of the kth layer by t_k, the total thickness of the laminated plate by $2h$. Then, we have

$$\sum_{k=1}^{N} t_k = 2h. \tag{62}$$

The "elastic constants" of the laminated plate treated as an inhomogeneous plate may be expressed by

$$c_{ij}(x_3) = c_{ij}^{(1)} H(x_3+h) + \sum_{k=1}^{N-1} (c_{ij}^{(k+1)} - c_{ij}^{(k)}) H\left(x_3 + h - \sum_{i=1}^{k} t_i \right), \tag{63}$$

where $H(x_3)$ denotes the Heaviside unit step function.

The expression can be used in the relevant expressions to determine $\tilde{Q}_{ij}$, the operators $\mathbf{L}_1$ and $\mathbf{L}_2$. Integration involving $c_{ij}(z)$ is straightforward. In particular, the coefficients A_{ij}, B_{ij} and D_{ij} in the equivalent CLT equations are given by

$$A_{ij} = \sum_{k=1}^{N} \tilde{Q}_{ij}^{(k)} (z_k - z_{k-1}), \quad B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \tilde{Q}_{ij}^{(k)} (z_k^2 - z_{k-1}^2), \quad D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \tilde{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3), \tag{64}$$

where $z_0 = -1$ and $z_N = 1$; z_k denote the dimensionless coordinates measured from the midplane to the interface of the $(k+1)$th and the kth layers.

These are exactly the same expressions found in the classical laminated plate theory. If the laminate is symmetric about the midplane, all B_{ij} vanish. Further, A_{16}, A_{26}, D_{16} and D_{26} are zero for a symmetric cross-ply laminated plate with orthotropic laminae.

6. THE GENERAL SOLUTION FOR SYMMETRIC PLATES

The equivalent CLT equations derived in the preceding sections are three simultaneous partial differential equations which are difficult to solve in general. We now consider the general solution for the symmetric plates. When the inhomogeneities and the geometry through thickness of the plate are symmetric about the midplane, (54), (55) and (57) are simplified to

$$\left(A_{11}\frac{\partial^2}{\partial x^2}+2A_{16}\frac{\partial^2}{\partial x\,\partial y}+A_{66}\frac{\partial^2}{\partial y^2}\right)u_0+\left[A_{16}\frac{\partial^2}{\partial x^2}+(A_{12}+A_{66})\frac{\partial^2}{\partial x\,\partial y}+A_{26}\frac{\partial^2}{\partial y^2}\right]v_0=0, \tag{65}$$

$$\left[A_{16}\frac{\partial^2}{\partial x^2}+(A_{12}+A_{66})\frac{\partial^2}{\partial x\,\partial y}+A_{26}\frac{\partial^2}{\partial y^2}\right]u_0+\left(A_{66}\frac{\partial^2}{\partial x^2}+2A_{26}\frac{\partial^2}{\partial x\,\partial y}+A_{22}\frac{\partial^2}{\partial y^2}\right)v_0=0, \tag{66}$$

$$\left[D_{11}\frac{\partial^4}{\partial x^4}+4D_{16}\frac{\partial^4}{\partial x^3\,\partial y}+2(D_{12}+2D_{66})\frac{\partial^4}{\partial x^2\,\partial y^2}+4D_{26}\frac{\partial^4}{\partial x\,\partial y^3}+D_{22}\frac{\partial^4}{\partial y^4}\right]w_0=\tilde{q}(x,y). \tag{67}$$

The general solution of (65) and (66) takes the form

$$u_0=f(x+sy), \tag{68}$$

$$v_0=h(x+sy). \tag{69}$$

Upon substituting f and h into (65) and (66), it can be shown that the nontrivial solution exists only if

$$l_1(s)l_3(s)-l_2^2(s)=0, \tag{70}$$

where

$$l_1(s)=A_{66}s^2+2A_{16}s+A_{11},$$
$$l_2(s)=A_{26}s^2+(A_{12}+A_{66})s+A_{16},$$
$$l_3(s)=A_{22}s^2+2A_{26}s+A_{66}.$$

The characteristic equation (70) gives four complex conjugate roots (Lekhnitskii, 1963). Let the complex roots with positive imaginary part be denoted by s_1 and s_2, then the general solutions to (65) and (66) are

$$u_0=2\,\mathrm{Re}\,\{f_1(\zeta_1)+f_2(\zeta_2)\}, \tag{71}$$

$$v_0=2\,\mathrm{Re}\,\{\lambda_1 f_1(\zeta_1)+\lambda_2 f_2(\zeta_2)\}, \tag{72}$$

where $\lambda_k=-l_1(s_k)/l_2(s_k)$, $\zeta_k=x+s_k y$, $(k=1,2)$; Re stands for the real part of a complex function.

When nonhomogeneous terms are present in (65) and (66), the corresponding particular solutions must be added to the general solution for the homogeneous equations.

Equation (67) is exactly the governing equation for bending of an anisotropic homogeneous plate (Lekhnitskii, 1968). The general solution for the homogeneous equation takes the form

$$w_0=2\,\mathrm{Re}\,\{g_1(\xi_1)+g_2(\xi_2)\}, \tag{73}$$

where $g_k(\xi_k)$ are analytic functions, and $\xi_k=x+\mu_k y$, $(k=1,2)$. The complex parameters μ_k are the roots of the characteristic equation

$$D_{22}\mu^4+4D_{26}\mu^3+2(D_{12}+2D_{66})\mu^2+4D_{16}\mu+D_{11}=0. \tag{74}$$

The general solution consists of (73) and a particular solution that is dependent on the loading function $\tilde{q}(x, y)$ of the problem.

Suppose for now that the analytic functions $f_k(\zeta_k)$ and $g_k(\xi_k)$ that satisfy the edge boundary conditions at the order ε^0 level have been determined. Upon substituting the relevant expressions into (58)–(61), we obtain, for order ε^2, the nonhomogeneous terms in (54) and (55) consisting of functions of $f_k^{(4)}(\zeta_k)$ and $g_k^{(5)}(\xi_k)$, and those in (57) consisting of functions of $f_k^{(5)}(\zeta_k)$ and $g_k^{(6)}(\zeta_k)$, where $f_k^{(n)}$, $g_k^{(n)}$ denote nth order differentiaton with respect to the argument. As a result, it can be easily shown that the solutions to order ε^2 equations, together with the ε^0 homogeneous solution, take the forms

$$u = 2\,\mathrm{Re}\left\{\sum_{k=1}^{2}[f_k(\zeta_k)+u_{2k}f_k''(\zeta_k)+u_{1k}g_k'(\xi_k)+u_{3k}g_k'''(\xi_k)]\right\}+O(\varepsilon^4), \tag{75}$$

$$v = 2\,\mathrm{Re}\left\{\sum_{k=1}^{2}[\lambda_k f_k(\zeta_k)+v_{2k}f_k''(\zeta_k)+v_{1k}g_k'(\xi_k)+v_{3k}g_k'''(\xi_k)]\right\}+O(\varepsilon^4), \tag{76}$$

$$w = 2\,\mathrm{Re}\left\{\sum_{k=1}^{2}[w_{1k}f_k'(\zeta_k)+g_k(\xi_k)+w_{2k}g_k''(\xi_k)]\right\}+O(\varepsilon^4), \tag{77}$$

$$\sigma_{xz} = 2\,\mathrm{Re}\left\{\sum_{k=1}^{2}[\sigma_{xz2}f_k''(\zeta_k)+\sigma_{xz4}f_k^{(4)}(\zeta_k)+\sigma_{xz3}g_k'''(\xi_k)+\sigma_{xz5}g_k^{(5)}(\xi_k)]\right\}+O(\varepsilon^4), \tag{78}$$

$$\sigma_{yz} = 2\,\mathrm{Re}\left\{\sum_{k=1}^{2}[\sigma_{yz2}f_k''(\zeta_k)+\sigma_{yz4}f_k^{(4)}(\zeta_k)+\sigma_{yz3}g_k'''(\xi_k)+\sigma_{yz5}g_k^{(5)}(\xi_k)]\right\}+O(\varepsilon^4), \tag{79}$$

$$\sigma_{z} = 2\,\mathrm{Re}\left\{\sum_{k=1}^{2}[\sigma_{z3}f_k'''(\zeta_k)+\sigma_{z5}f_k^{(5)}(\zeta_k)+\sigma_{z4}g_k^{(4)}(\xi_k)+\sigma_{z6}g_k^{(6)}(\xi_k)]\right\}+O(\varepsilon^4), \tag{80}$$

in which the coefficients u_{1k}, u_{2k}, u_{3k}, $v_{1k}, \ldots, \sigma_{z6}$ are functions of z and are related to the integrals involving $c_{ij}(z)$.

We note that in Basi *et al.* (1991) similar forms of stretching mode solution and bending mode solution valid for the case of zero lateral traction were given. Here we arrive at the expressions for this special case in a clear and systematic manner. The solution forms (75)–(80) can be used in conjunction with the complex variable approach to solve the bending and stretching problems of the "equivalent" homogeneous anisotropic plates.

7. ILLUSTRATIVE EXAMPLES

(1) *Anisotropic inhomogeneous plates under forces and moments distributed uniformly along the edges*

A rectangular plate is deformed under forces and moments per unit length distributed uniformly along the edges. On the lateral surfaces the plate is free from tractions. To illustrate the basic theory, we assume that the plate is symmetric about the midplane.

At the ε^0 level, the analytic functions in (71)–(73) may be obtained by letting

$$f_k(\zeta_k) = a_k\zeta_k, \tag{81}$$

$$g_k(\xi_k) = \tfrac{1}{2}b_k\xi_k^2+c_k\xi_k, \tag{82}$$

where a_k, b_k and c_k $(k = 1, 2)$ are complex coefficients to be determined from the edge conditions.

Let us restrict attention to the interior solution and disregard the boundary layer effect for simplicity. Then the edge conditions can be easily satisfied in an average sense by considering the resultant forces and moments through the thickness.

The averaged edge conditions for the ε^0 solution are given by:

On $x = 0, a$:

$$\int_{-1}^{1} \begin{bmatrix} l_{14} & l_{24} \\ l_{16} & l_{26} \end{bmatrix} \left(\begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} - z \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} w_0 \right) \mathrm{d}z = \begin{Bmatrix} N_x \\ N_{xy} \end{Bmatrix}, \tag{83}$$

$$\int_{-1}^{1} \begin{bmatrix} l_{14} & l_{24} \\ l_{16} & l_{26} \end{bmatrix} \left(\begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} - z \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} w_0 \right) z \, \mathrm{d}z = \begin{Bmatrix} M_x \\ M_{xy} \end{Bmatrix}. \tag{84}$$

On $y = 0, b$:

$$\int_{-1}^{1} \begin{bmatrix} l_{15} & l_{25} \\ l_{16} & l_{26} \end{bmatrix} \left(\begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} - z \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} w_0 \right) \mathrm{d}z = \begin{Bmatrix} N_y \\ N_{xy} \end{Bmatrix}, \tag{85}$$

$$\int_{-1}^{1} \begin{bmatrix} l_{15} & l_{25} \\ l_{16} & l_{26} \end{bmatrix} \left(\begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} - z \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} w_0 \right) z \, \mathrm{d}z = \begin{Bmatrix} M_y \\ M_{xy} \end{Bmatrix}. \tag{86}$$

where N_x, N_y and N_{xy} are the dimensionless resultant uniform tractions through the thickness; M_x, M_y and M_{xy} are the dimensionless uniform moments acting along the four edges, respectively.

Use of (81) and (82) in (83)–(86) provides the following six independent algebraic equations to determine eight unknowns in the complex coefficients a_k and b_k, of which two are arbitrary constants characterizing a rigid body motion:

$$2\,\mathrm{Re} \left\{ \sum_{k=1}^{2} \begin{bmatrix} A_{11}+s_k A_{16} & A_{16}+s_k A_{12} \\ A_{12}+s_k A_{26} & A_{26}+s_k A_{22} \\ A_{16}+s_k A_{66} & A_{66}+s_k A_{26} \end{bmatrix} \begin{Bmatrix} a_k \\ \lambda_k a_k \end{Bmatrix} \right\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}, \tag{87}$$

$$-2\,\mathrm{Re} \left\{ \sum_{k=1}^{2} \begin{bmatrix} D_{11}+\mu_k D_{16} & D_{16}+\mu_k D_{12} \\ D_{12}+\mu_k D_{26} & D_{26}+\mu_k D_{22} \\ D_{16}+\mu_k D_{66} & D_{66}+\mu_k D_{26} \end{bmatrix} \begin{Bmatrix} b_k \\ \mu_k b_k \end{Bmatrix} \right\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}. \tag{88}$$

The terms corresponding to $c_k \xi_k$ in (82) represent a rigid body rotation of the plate. The coefficients c_k are arbitrary and may be so chosen as to make the slope at the origin zero.

Continuing the solution to order ε^2, we find immediately that the governing equations and the edge conditions at ε^2-order level are all homogeneous. Therefore, the solution is simply $u_1 = v_1 = w_1 = 0$, and the ε^2 solution gives correction only to w. No further correction to the solution is needed afterwards. The analytic solution for the problem is given by combining the ε^0 solution with the ε^2 correction as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = 2\,\mathrm{Re}\left\{\sum_{k=1}^{2}\left(\begin{Bmatrix} a_k\zeta_k \\ \lambda_k a_k \zeta_k \end{Bmatrix} - z\begin{Bmatrix} b_k\xi_k \\ \mu_k b_k \xi_k \end{Bmatrix}\right)\right\}, \tag{89}$$

$$w = \mathrm{Re}\left\{\sum_{k=1}^{2}\left[b_k\xi_k^2 + 2\varepsilon^2\int_{-1}^{z}\left([l_{34}+s_k l_{36} l_{36} + s_k l_{35}]\begin{Bmatrix} a_k \\ \lambda_k a_k \end{Bmatrix} - z[l_{34}+\mu_k l_{36} l_{36} + \mu_k l_{35}]\begin{Bmatrix} b_k \\ \mu_k b_k \end{Bmatrix}\right)\mathrm{d}z\right]\right\}, \tag{90}$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_z = 0, \tag{91}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = 2\,\mathrm{Re}\left\{\sum_{k=1}^{2}\left(\begin{bmatrix} \tilde{Q}_{11}+s_k\tilde{Q}_{16} & \tilde{Q}_{16}+s_k\tilde{Q}_{12} \\ \tilde{Q}_{12}+s_k\tilde{Q}_{26} & \tilde{Q}_{26}+s_k Q_{22} \\ \tilde{Q}_{16}+s_k\tilde{Q}_{66} & \tilde{Q}_{66}+s_k\tilde{Q}_{26} \end{bmatrix}\begin{Bmatrix} a_k \\ \lambda_k a_k \end{Bmatrix} - z\begin{bmatrix} \tilde{Q}_{11}+\mu_k\tilde{Q}_{16} & \tilde{Q}_{16}+\mu_k\tilde{Q}_{12} \\ \tilde{Q}_{12}+\mu_k\tilde{Q}_{26} & \tilde{Q}_{26}+\mu_k\tilde{Q}_{22} \\ \tilde{Q}_{16}+\mu_k\tilde{Q}_{66} & \tilde{Q}_{66}+\mu_k\tilde{Q}_{26} \end{bmatrix}\begin{Bmatrix} b_k \\ \mu_k b_k \end{Bmatrix}\right)\right\}. \tag{92}$$

The foregoing solution is valid everywhere except at points in the boundary layer along the edges. The effect of the boundary layer needs to be assessed, especially at the free edge. However, it is beyond the scope of the present work.

(2) *Pagano's problem*

The benchmark problem solved by Pagano (Pagano, 1970) is now studied in the context of the inhomogeneous plate. We assume that the field variables have been made dimensionless in the analysis.

Considering the bending problem of a cross-ply symmetric plate with orthotropic layers, under the action of lateral load that is represented by a double Fourier series in the form

$$\tilde{q}(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\alpha_m x\sin\beta_n y, \tag{93}$$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$. For brevity we shall drop the double summation signs in the solution.

The simple support conditions on the four edges of the plate are given by

$$\sigma_x = v = w = 0 \quad \text{on } x = 0, a,$$
$$\sigma_y = u = w = 0 \quad \text{on } y = 0, b. \tag{94}$$

With $B_{ij} = A_{16} = A_{26} = D_{16} = D_{26} = 0$ in the formulation, the ε^0 order equations are solved by letting

$$u_0 = v_0 = 0, \quad w_0 = k_0\sin\alpha_m x\sin\beta_n y. \tag{95}$$

Substituting (95) into (57), we find the coefficient k_0 to be

$$k_0 = q_{mn}/[D_{11}\alpha_m^4 + 2(D_{12}+2D_{66})\alpha_m^2\beta_n^2 + D_{22}\beta_n^4].$$

Upon using (95) in (49), (50) and (27), we obtain the ε^0-order solution as follows:

$$u_{(0)} = -\alpha_m k_0 z \cos \alpha_m x \sin \beta_n y, \tag{96}$$

$$v_{(0)} = -\beta_n k_0 z \sin \alpha_m x \cos \beta_n y, \tag{97}$$

$$w_{(0)} = k_0 \sin \alpha_m x \sin \beta_n y, \tag{98}$$

$$\sigma_{xz(0)} = k_0 \sigma_{xz0}(z) \cos \alpha_m x \sin \beta_n y, \tag{99}$$

$$\sigma_{yz(0)} = k_0 \sigma_{yz0}(z) \sin \alpha_m x \cos \beta_n y, \tag{100}$$

$$\sigma_{z(0)} = [k_0 \sigma_{z0}(z) - q_{mn}] \sin \alpha_m x \sin \beta_n y, \tag{101}$$

$$\sigma_{x(0)} = k_0 z(\alpha_m^2 \tilde{Q}_{11} + \beta_n^2 \tilde{Q}_{12}) \sin \alpha_m x \sin \beta_n y, \tag{102}$$

$$\sigma_{y(0)} = k_0 z(\alpha_m^2 \tilde{Q}_{12} + \beta_n^2 \tilde{Q}_{22}) \sin \alpha_m x \sin \beta_n y, \tag{103}$$

$$\sigma_{xy(0)} = -2k_0 z \alpha_m \beta_n \tilde{Q}_{66} \cos \alpha_m x \cos \beta_n y, \tag{104}$$

where

$$\begin{Bmatrix} \sigma_{xz0}(z) \\ \sigma_{yz0}(z) \end{Bmatrix} = -\int_{-1}^{z} \begin{Bmatrix} \alpha_m^3 \tilde{Q}_{11} + \alpha_m \beta_n^2 (\tilde{Q}_{12} + 2\tilde{Q}_{66}) \\ \alpha_m^2 \beta_n (\tilde{Q}_{12} + 2\tilde{Q}_{66}) + \beta_n^3 \tilde{Q}_{22} \end{Bmatrix} \eta \, d\eta,$$

$$\sigma_{z0}(z) = -\int_{-1}^{z} (z-\eta)[\alpha_m^4 \tilde{Q}_{11} + 2\alpha_m^2 \beta_n^2 (\tilde{Q}_{12} + 2\tilde{Q}_{66}) + \beta_n^4 \tilde{Q}_{22}] \eta \, d\eta.$$

Carrying on the solution to order ε^2, we find that the nonhomogeneous terms in the ε^2 equations are

$$f_{11}(x, y, 1) = k_1 \cos \alpha_m x \sin \beta_n y,$$
$$f_{21}(x, y, 1) = k_2 \sin \alpha_m x \cos \beta_n y,$$
$$\tilde{q}_1 = k_3 \sin \alpha_m x \sin \beta_n y,$$

where k_i ($i = 1, 2, 3$) and the relevant functions for the ε^2 solution are given in the Appendix.

The ε^2 solution can be obtained by letting

$$u_1 = U_1 \cos \alpha_m x \sin \beta_n y, \tag{105}$$

$$v_1 = U_2 \sin \alpha_m x \cos \beta_n y, \tag{106}$$

$$w_1 = U_3 \sin \alpha_m x \sin \beta_n y. \tag{107}$$

Determination of the second-order corrections is straightforward afterwards. In view of the recurrence of the solution forms, the asymptotic solution for the problem can be determined to any order and provides results with a high degree of accuracy. We shall not list the lengthy analytical expressions here for brevity. To demonstrate the performance of the asymptotic solution, we consider a rectangular [0/90/0] laminated plate under sinusoidal lateral load ($m = n = 1$), with $b = 3a$ and the layer material properties given by $E_L/E_T = 25$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.2$, $\nu_{LT} = \nu_{TT} = 0.25$, where the subscripts L and T refer to the longitudinal direction and transverse to fiber direction. The elastic constants can be deduced from these data. In Table 1 the numerical results for the transverse displacement and the stresses, in comparison with Pagano's elasticity solution as well as the results obtained according to CLT, the first-order shear deformation theory (FSDT) and a higher-order shear deformation theory (HSDT) (Reddy, 1984), at several selected locations as indicated in the table, are given. The transverse displacement and the stresses in Table 1 are normalized

Table 1. Comparisons of the results for benchmark problem according to various theories

a/h	Theories	$\bar{w}$ $(a/2, b/2, 0)$	τ_x $(a/2, b/2, 1/2)$	τ_y $(a/2, b/2, 1/6)$	τ_{xy} $(0, 0, 1/2)$	τ_{xz} $(0, b/2, 0)$	τ_{yz} $(a/2, 0, 0)$	τ_z $(a/2, b/2, 0)$
	Pagano	2.8200	1.1400	0.1090	−0.0269	0.3510	0.0334	*****
	CLT	0.5030	0.6230	0.0252	−0.0083	0.4400	0.0108	*****
	FSDT	2.3626	0.6130	0.0934	−0.0205	0.4357	0.0282	0.5000
4	HSDT	2.6411	1.0359	0.1028	−0.0263	0.3825	0.0304	0.5000
	Present (ε^0)	0.5034	0.6233	0.0251	−0.0083	0.4395	0.0108	0.5000
	Present (ε^2)	3.1778	1.3004	0.1344	−0.0326	0.3091	0.0400	0.5025
	Present (ε^4)	2.6584	1.0622	0.0988	−0.0244	0.3715	0.0309	0.4966
	Pagano	0.9190	0.7260	0.0418	−0.0120	0.4200	0.0152	*****
	CLT	0.5030	0.6230	0.0252	−0.0083	0.4400	0.0108	*****
	FSDT	0.8030	0.6214	0.0375	−0.0115	0.4388	0.0139	0.5000
10	HSDT	0.8622	0.6924	0.0398	−0.0115	0.4299	0.0145	0.5000
	Present (ε^0)	0.5034	0.6233	0.0251	−0.0083	0.4395	0.0108	0.5000
	Present (ε^2)	0.9313	0.7316	0.0426	−0.0122	0.4187	0.0155	0.5004
	Present (ε^4)	0.9180	0.7255	0.0417	−0.0120	0.4202	0.0152	0.5002
	Pagano	0.6100	0.6500	0.0299	−0.0093	0.4340	0.0119	*****
	CLT	0.5030	0.6230	0.0252	−0.0083	0.4400	0.0108	*****
	FSDT	0.5784	0.6228	0.0283	−0.0088	0.4393	0.0116	0.5000
20	HSDT	0.5937	0.6407	0.0289	−0.0091	0.4371	0.0117	0.5000
	Present (ε^0)	0.5034	0.6233	0.0251	−0.0083	0.4395	0.0108	0.5000
	Present (ε^2)	0.6104	0.6504	0.0295	−0.0093	0.4343	0.0120	0.5001
	Present (ε^4)	0.6095	0.6500	0.0295	−0.0092	0.4344	0.0119	0.5001

as $\bar{w} = 100E_T w/q_{11}hS^4$, $[\tau_x, \tau_y, \tau_{xy}] = [\sigma_x, \sigma_y, \sigma_{xy}]/q_{11}S^2$, $[\tau_{xz}, \tau_{yz}] = [\sigma_{xz}, \sigma_{yz}]/q_{11}S$, $\tau_z = \sigma_z/q_{11}$ and $S = a/h$. In cases where the span-to-thickness ratio a/h is large, the second-order modifications are minor. Indeed the asymptotic analysis works best when ε is small. At the ε^2 level, the results are practically identical to the elasticity solution when $a/h > 20$ ($\varepsilon < 0.05$). As the thickness increases, the higher order modifications become significant. When $a/h < 10$, it is necessary to carry out the analysis to ε^4-order to obtain accurate results. Compared with those obtained from other theories, the ε^4 results are quite acceptable even in the case of a very thick plate ($a/h = 4$) in which the thickness effect is very pronounced.

8. CONCLUSIONS

Based on the three-dimensional elasticity without *a priori* assumptions, we have developed an asymptotic theory for bending and stretching of anisotropic inhomogeneous and laminated plates under the action of lateral tractions and edge loads. Through non-dimensionalization of the basic equations and expansion of the field variables in powers of ε^2, we showed by successive integration that the governing equations at each level of the asymptotic solution are precisely the CLT equations with nonhomogeneous terms for bending and stretching of anisotropic homogeneous plates. The boundary conditions on the lateral surfaces of the plate are exactly satisfied in the formulation. The appropriate edge boundary conditions for a problem are clearly specified. While the solution can be obtained in a systematic way no more difficult than the CLT solution for the problem, the asymptotic solution converges to the accurate results according to the power of ε^2. The theory has been illustrated by determining the elasticity solutions for benchmark problems. Analytic solutions were obtained without treating the interfacial continuity individually in the case of laminated plates. Extension of the theory to include thermal effect can be readily made. Consideration of the dynamic responses of the anisotropic inhomogeneous and laminated plates is presented in another paper (Tarn and Wang, 1994).

Acknowledgement—This work is supported by the National Science Council of R.O.C. through grant NSC83-0410-E-006-040.

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APPENDIX

The relevant functions for the ε^2 solution of the problem are

$$\varphi_1 = \tilde{\varphi}_1(z)k_0 \sin \alpha_m x \sin \beta_n y, \tag{A1}$$

$$\phi_{11} = \tilde{\phi}_{11}(z)k_0 \cos \alpha_m x \sin \beta_n y, \tag{A2}$$

$$\phi_{21} = \tilde{\Phi}_{21}(z)k_0 \sin \alpha_m x \cos \beta_n y, \tag{A3}$$

$$f_{11} = \bar{k}_1(z)k_0 \cos \alpha_m x \sin \beta_n y, \tag{A4}$$

$$f_{21} = \bar{k}_2(z)k_0 \sin \alpha_m x \cos \beta_n y, \tag{A5}$$

$$q_1 = \bar{k}_3(z)k_0 \sin \alpha_m x \sin \beta_n y, \tag{A6}$$

in which

$$\tilde{\varphi}_1(z) = -\int_{-1}^{z} (c_{13}\alpha_m^2 + c_{23}\beta_n^2)c_{33}^{-1}\eta \, d\eta,$$

$$\tilde{\Phi}_{11}(z) = \int_{-1}^{z} [\tilde{s}_{55}\sigma_{xz0} + (z-\eta)(c_{13}\alpha_m^2 + c_{23}\beta_n^2)c_{33}^{-1}\alpha_m\eta] \, d\eta,$$

$$\tilde{\Phi}_{21}(z) = \int_{-1}^{z} [\tilde{s}_{44}\sigma_{yz0} + (z-\eta)(c_{13}\alpha_m^2 + c_{23}\beta_n^2)c_{33}^{-1}\beta_n\eta] \, d\eta,$$

$$\bar{f}_{11}(z) = (\tilde{Q}_{11}\alpha_m^2 + \tilde{Q}_{66}\beta_n^2)\tilde{\Phi}_{11} + (\tilde{Q}_{12} + \tilde{Q}_{66})\alpha_m\beta_n\tilde{\Phi}_{21} - \alpha_m\sigma_{z0}c_{13}c_{33}^{-1},$$

$$\bar{f}_{21}(z) = (\tilde{Q}_{12} + \tilde{Q}_{66})\alpha_m\beta_n\tilde{\Phi}_{11} + (\tilde{Q}_{66}\alpha_m^2 + \tilde{Q}_{22}\beta_n^2)\tilde{\Phi}_{21} - \beta_n\sigma_{z0}c_{23}c_{33}^{-1},$$

$$\bar{k}_1(z) = \int_{-1}^{z} \bar{f}_{11}(\eta) \, d\eta, \qquad \bar{k}_2(z) = \int_{-1}^{z} \bar{f}_{21}(\eta) \, d\eta,$$

$$\bar{k}_3(z) = \int_{-1}^{z} (z-\eta)(\alpha_m \bar{f}_{11} + \beta_n \bar{f}_{21}) \, d\eta.$$

At $z = 1$ we have

$$k_1 = \int_{-1}^{1} \bar{f}_{11}(z) \, dz, \qquad k_2 = \int_{-1}^{1} \bar{f}_{21}(z) \, dz, \tag{A7, 8}$$

$$k_3 = \int_{-1}^{1} (1-z)(\alpha_m \bar{f}_{11} + \beta_n \bar{f}_{21}) \, dz. \tag{A9}$$